

# DIFFERENTIAL GALOIS THEORY AND FINITE GENERATION OF FIBER RINGS IN SEMPLE TOWER

MOHAMMAD REZA RAHMATI

ABSTRACT. In this short note we briefly discuss the finite generation of fiber rings of invariant  $k$ -jets of holomorphic curves in a complex projective manifold, using differential Galois theory.

## 1. INTRODUCTION

The action of the local automorphism group  $G_k$  of  $(\mathbb{C}, 0)$  on the bundle  $J_k T^*X$  of  $k$ -jets of holomorphic curves  $f : \mathbb{C} \rightarrow X$  in a complex projective manifold  $X$  has been a focus of investigation since the work of Green and Griffiths [GG], developed by Demaily [D], Diverio, Merker and Rousseau in [DMR]. The problem to determine the finite generation of complex graded algebras  $\mathcal{O}(J_{k,x})^{G_k}$  has been a long time an open problem in this area. The invariant jets play an important role in the strategy introduced by Green-Griffiths [GG], Bloch [B], Demaily [D] in order to prove Kobayashi hyperbolicity conjecture [Ko].

In general one may work with a holomorphic subbundle  $V$  of the tangent bundle  $T(X)$ . However for what is important in this text the two context are equal. Green and Griffiths [GG] introduce the vector bundle

$$(1) \quad E_{k,m}^{GG} V^* \rightarrow X$$

whose fibers are polynomials  $Q(z; \xi_1, \dots, \xi_r)$  of weighted degree  $m$ , where  $z$  is a local variable on  $X$  and  $\xi_i$  may be thought as a variable on the fibers of  $X_i \rightarrow X_{i-1}$ . One can identify  $E_{k,m}^{GG} V^*$  with the bundle of germs of  $k$ -jets of the analytic maps  $f : (\mathbb{C}, 0) \rightarrow (X, x)$  tangent to  $V$ , denoted as  $J_k V$ . The weights attached to  $\xi_1, \dots, \xi_k, \dots$  should be  $1, 2, \dots, k, \dots$  respectively and the total weighted degree of a homogeneous germ is defined as

$$(2) \quad Q(z, \xi) = \sum_{\alpha} A_{\alpha}(z) \xi_1^{\alpha_1} \dots \xi_k^{\alpha_k}, \quad |\alpha| = \alpha_1 + 2\alpha_2 + \dots + k\alpha_k$$

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*Key words and phrases.* Demaily-Semple tower, Jets of entire curves, Differential Galois group, Differential field, Wronskian, Ring of invariant jets.

There is a natural  $\mathbb{C}^*$ -action on the space of germs of holomorphic maps  $f : \mathbb{C} \rightarrow X$  according to the above weights. Then

$$(3) \quad E_{k,m}^{GG} V^* = J_k V / \mathbb{C}^*$$

The symbol  $V^*$  used in  $E_{k,m}^{GG} V^*$  is only a notation reminding the duality appearing in the fact that  $S^* V = Sym^{\otimes} V$  can be understood as polynomial algebra on a basis of  $V^*$ . The analytic structure on  $E_{k,m}^{GG} V^*$  comes from the fact that, one can arrange the germs to depend holomorphically on their initial values at 0. J. P. Demailly [D2] develops the ideas in [GG] and considers the jets of differentials that are also invariant under change of coordinate on  $\mathbb{C}$ . In fact the change of coordinates on the manifold  $\mathbb{C}$  around 0 affects the  $k$ -th derivative by

$$(4) \quad (f \circ \psi)^{(k)}(0) = \psi'(0).f^{(k)} + \text{higher order terms}$$

If we denote the group of all these automorphisms by  $G'_k$ , then the total group of automorphisms of  $J_k V$  we quotient by is  $G_k = \mathbb{C}^* \ltimes G'_k$ . The question of the finite generation of the ring of fibers of  $(J_k V)^{G_k}$  raise up from the non-reductive subgroups in  $G'_k$  that are included in the unipotent radical  $U_k = G'_k$ .

In this note we provide some answer to this using differential Galois theory. To us a differential field is a field  $F$  with a derivation

$$(5) \quad \delta : F \rightarrow F$$

The subfield of elements where  $\delta$  vanishes is called the field of constants. A differential automorphism is one which commutes with  $\delta$ . Differential Galois theory studies the formalism of differential field extensions as well as their differential Galois groups. The result is that, a similar theorem to the main theorem of Galois theory for usual fields can be stated in this case. However the differential Galois groups are algebraic matrix groups. We pose the analysis of the finite generation of the ring of invariant jets into a similar problem on the field quotients. The operation of formal differentiation equips these fields a differential field. This will help us to apply the main theorem of differential Galois extensions to deduce that the field of invariant jets is finite algebraic over the the field generated by Wronskians.

The text is divided in two parts. In the first part we explain the proof by G. Berczi and F. Kirwan [BK] on the finite generation of invariant jets of curves pointwise. Our purpose in the first section is to introduce the problem more clear and follow the text in the reference. In the next Section we use the tools from the differential Galois theory [K] to answer the same question.

## 2. GENERALIZED DEMAILY-SEMPLE BUNDLES

Let  $X$  be a complex  $n$ -dimensional projective manifold. In [GG] Green and Griffiths introduce the bundle  $J_k \rightarrow X$  of germs of parametrized curves in  $X$ . Its fiber at  $x \in X$  is the set of equivalence classes of germs of holomorphic maps  $f : (\mathbb{C}, 0) \rightarrow (X, x)$  with equivalence relation  $f^{(j)}(0) = g^{(j)}(0)$ ,  $0 \leq j \leq k$ . By choosing local holomorphic coordinates around  $x$ , the elements of the fiber  $J_{k,x}$  can be represented by the Taylor expansion

$$(6) \quad f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \dots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

Setting  $f = (f_1, \dots, f_n)$  on open neighborhoods of  $0 \in \mathbb{C}$ , the fiber is

$$(7) \quad J_{k,x} = \{(f'(0), \dots, f^{(k)}(0))\} = \mathbb{C}^{nk}$$

Lets  $G_k$  be the group of local reparametrizations of  $(\mathbb{C}, 0)$

$$(8) \quad t \mapsto \phi(t) = \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*$$

Its action on the  $k$ -jets is given by the following matrix multiplication

$$(9) \quad [f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!]. \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ 0 & a_1^2 & 2a_1a_2 & \dots & a_1a_{k-1} + \dots a_{k-1}a_1 \\ 0 & 0 & a_1^3 & \dots & 3a_1^2a_{k-2} + \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_1^k \end{bmatrix}$$

The unipotent elements correspond to the matrices with  $a_1 = 1$ . There is a short exact sequence

$$(10) \quad 1 \rightarrow U_k \rightarrow G_k \rightarrow \mathbb{C}^* \rightarrow 0$$

where  $U_k$  is the unipotent radical of  $G_k$ . The action of  $\mathbb{C}^*$  on  $k$ -jets is

$$(11) \quad \lambda.(f'(0), \dots, f^{(k)}(0)) = (\lambda.f'(0), \dots, \lambda^k.f^{(k)}(0))$$

Let  $E_{k,m}$  be the Demaily-Semple bundle whose fiber at  $x$  consists of  $U_k$ -invariant polynomials on the fiber coordinates of  $J_k$  at  $x$  of weighted degree  $m$ . Set  $E_k = \bigoplus_m E_{k,m}$ , the Demaily-Semple bundle of graded algebras of invariants.

The aforementioned construction can be generalized by considering the bundle  $J_{k,p} \rightarrow X$  of  $k$ -jets of maps  $f : \mathbb{C}^p \rightarrow X$ . It is the germ of equivalence classes of Laurent series (1) but now the variable  $t$  is a vector variable, and

$$(12) \quad f^{(j)}(0) \in \text{Hom}(\text{Sym}^j \mathbb{C}^p, \mathbb{C}^n), \quad J_{k,p,x} = \{(f'(0), \dots, f^{(k)}(0))\} = \mathbb{C}^n \binom{k+p-1}{k-1}$$

The analogues of the groups  $G_k$  and its unipotent radical as well as the bundle  $E_k$  can be defined similarly in this case, denoted by  $G_{k,p}$ ,  $U_{k,p}$ ,  $E_{k,p}$ . In this case the action of the automorphisms of  $(\mathbb{C}^p, 0)$  is given by the same matrix as in (9), however the entries  $a_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$  are a  $p \times \dim(\text{Sym}^i \mathbb{C}^p)$  matrix and

$$(13) \quad \sum_{\sigma \in S_l} a_{i_1} \otimes \dots \otimes a_{i_l} : (\mathbb{C}^p)^{\otimes i_1} \otimes \dots \otimes (\mathbb{C}^p)^{\otimes i_l} \rightarrow (\mathbb{C}^p)^{\otimes l}$$

We summarize these in the following definition.

**Definition 2.1.** *The generalized Demaily-Semple bundle  $E_{k,m,p} \rightarrow X$  has fibers consisting of the  $G_{k,p}$ -invariant jet differentials of order  $k$  and weighted degrees  $(m, m, \dots, m)$ , that is the complex valued polynomials  $Q(f'(0), f''(0), \dots, f^{(k)}(0))$  on the fibers of  $J_{k,p}$  which transform under any reparametrization  $\phi \in G_{k,p}$  of  $(\mathbb{C}^p, 0)$  as*

$$(14) \quad Q(f \circ \phi) = J_\phi^m Q(f) \circ \phi$$

where  $J_\phi = \det(\Phi)$  denotes the jacobian of  $\phi$  at 0. The generalized Demaily-Semple bundle of algebras  $E_{k,p} = \oplus_m E_{k,p,m}$  is the associated graded algebra of  $G'_{k,p}$ -invariants, whose fiber at  $x$  is the generalized Demaily-Semple algebra  $\mathcal{O}(J_{k,p})_x^{G'_{k,p}}$ .

The group  $G_{k,p}$  sits into the extension

$$(15) \quad 1 \rightarrow U_{k,p} \rightarrow G_{k,p} \rightarrow GL(p) \rightarrow 0, \quad 1 \rightarrow U_{k,p} \rightarrow G'_{k,p} \rightarrow SL(p) \rightarrow 0$$

where  $G'_{k,p} = G_{k,p}/GL(p)$ . We propose to sketch the machinery in [BK] to make the set up more clear. We express this in case  $p = 1$ , with the understanding that the generalized case can be carried word by word the similarly. First we call a  $k$ -jet  $f : \mathbb{C} \rightarrow X$  regular if  $f'(0) \neq 0$ . We call such a jet non-degenerate when it has maximal rank in local coordinates. The methodology in [BK] is based on the standard embedding

$$(16) \quad \Phi : J_k^{\text{reg}}/G_k \hookrightarrow \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \text{Sym}^{\leq k} \mathbb{C}^n),$$

where the second embedding is the Plucker embedding, and

$$(17) \quad \Psi : J_k^{\text{reg}}/G_k \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

of flags of length  $k$ . Let  $e_1, \dots, e_n$  be standard basis of  $\mathbb{C}^n$ . Then a basis of  $\text{Sym}^{\leq k} \mathbb{C}^n$  consists of

$$(18) \quad \{e_{i_1 \dots i_s} = e_{i_1} \dots e_{i_s}, s \leq k\}$$

and a basis of  $\mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n)$  is

$$(19) \quad \{e_{i_1} \wedge \dots \wedge e_{i_s}, s \leq k\}$$

The corresponding coordinates of  $x \in \text{Sym}^{\leq k} \mathbb{C}^n$  will be denoted by  $x_{i_1 \dots i_s}$ . Assume  $X_{n,k}$  be the image of non-degenerate  $k$ -jets. Write

$$(20) \quad z = \Phi(e_1, \dots, e_k) = [e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum e_{i_1} \dots e_{i_k})]$$

The group  $Gl(n)$  acts on  $z \in X_{n,k}$  as follows. If  $g \in Gl(n)$  has the column vectors  $v_1, \dots, v_n$ , then

$$(21) \quad g.z = \Phi(v_1, \dots, v_n) = [v_1 \wedge (v_2 + v_1^2) \wedge \dots \wedge (\sum v_{i_1} \dots v_{i_k})]$$

With this settings one can show that if  $n \geq k$  then  $X_{n,k}$  is the  $Gl(n)$  orbit of  $z$  above. In this case the image of regular  $k$ -jets is a finite union of  $Gl(n)$ -orbits, denoted  $Y_{n,k}$ . If  $k > n$  then  $X_{n,k}$  and  $Y_{n,k}$  are  $Gl(n)$ -invariant quasi-projective varieties, with no dense  $Gl(n)$ -orbits, [BK] page 17.

**Proposition 2.2.** *If  $k \geq 2$  then  $U_k$  is a Grosshans subgroup of  $SL(k)$ , so that any linear action of  $U_k$  which extends to a linear action of  $SL(k)$  has finitely generated invariants.*

By a Grosshans subgroup we mean the following: If there exists a complex reductive group  $G$  containing the unipotent radical  $U$  such that  $\mathcal{O}(G)^U$  is finitely generated and the action of  $U$  on  $X$  extends to the action of  $G$ , then

$$(22) \quad \mathcal{O}(J_{k,x})^U = (\mathcal{O}(J_{k,x}) \otimes \mathcal{O}(G)^U)^G$$

is finitely generated.

**Corollary 2.3.** *The fibers  $\mathcal{O}(J_k)_x^{U_k}$  and  $\mathcal{O}(J_{k,p})_x^{G'_{k,p}}$  of the bundles  $E_k$  and  $E_{k,p}$  are finitely generated graded complex algebras.*

In the Grosshans situation one easily can deduce that  $\mathcal{O}(J_{k,x})^{G'_k} = (\mathcal{O}(J_{k,x})^U)^{G'_k/U}$  is finitely generated. A more specific result in [BK] is the following;

**Proposition 2.4.** [BK] *The group  $G'_{k,p}$  is a subgroup of the special linear group  $SL(\text{sym}^{\leq k} \mathbb{C}^p)$  such that the algebra of invariants  $\mathcal{O}(SL(\text{sym}^{\leq k} \mathbb{C}^p))^{G'_{k,p}}$  is finitely generated and every linear action of  $G'_{k,p}$  or  $G_{k,p}$  on an affine or projective variety (with an ample linearization) which extends to a linear action of  $GL(\text{sym}^{\leq k} \mathbb{C}^n)$  has finitely generated invariants.*

In this text we reprove this fact by the Galois theory of differential fields. In fact the whole of the next section is the proof of this.

**Remark 2.5.** *In the situation of Green-Griffiths bundles one sets  $E_{k,\bullet}^{GG} = \bigoplus E_{k,m}^{GG}$ . Define*

$$(23) \quad W_k E_{\bullet,\bullet}^{GG} = \bigcup_{l=1}^k E_{l,\bullet}^{GG}$$

*Then  $W_k$  constitute an increasing filtration of vector bundles on  $X$  of type  $(r - 1, \dots, r - 1)$ . The space  $\text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$  can be regarded as a kind of moduli for all the flags of the same type as  $W_k$ . The group  $G_k$  is an algebraic group preserving these flags. The aforementioned construction is similar to that of flag manifolds parametrizing filtrations of the same kind on a vector space.*

### 3. FORMALISM OF DIFFERENTIAL FIELDS

The rings we are considering in this section are all commutative with 1. A derivation of the ring  $A$  is an additive mapping

$$(24) \quad \delta : A \rightarrow A, \quad \delta(ab) = b\delta(a) + a\delta(b)$$

If  $A$  is an integral domain, derivation of  $A$  uniquely extends to the quotient field of  $A$ . A differential ring is a pair  $(A, \delta)$  as above. The operation  $\delta$  often denoted by  $(.)'$  as in  $x', x'', \dots$ . Then the elements in such a ring can be considered as polynomials in  $x$  and  $\delta(x)$  for  $x \in A$  which can be chosen among the generators of  $A$ . An ideal  $I$  in  $A$  which  $I' \subset I$  is called a differential ideal. A differential homomorphism is one which commutes with the derivations. An isomorphism between two fields is called admissible if there exists a field containing both of them.

We propose to set a Galois theory for differential fields. Let  $A$  be a differential field and  $B$  a differential subfield. The differential Galois group  $G$  of  $A/B$  is the group of all differential automorphisms of  $A$  living  $B$  fixed. Then the same formalism like the Galois groups of usual fields appear here also. For any intermediate differential subfield  $C$ , denote the subgroup of  $G$  living  $C$  elementwise fixed by  $C'$ ; and similar for any subgroup  $H$  of  $G$  denote by  $H'$  the elements in  $A$  fixed by that. Call a field or group closed if it is equal to its double prime. Now with these notations making PRIMED defines the Galois correspondence between closed subgroups and closed differential subfields.

Let  $A$  be a differential field of characteristic 0 with an algebraically closed constant field. The Wronskian of  $n$  elements  $y_1, \dots, y_n$  in a differential ring is defined as the determinant

$$(25) \quad \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ & & & \\ & & & \\ y_1^{(n)} & & & y_n^{(n-1)} \end{vmatrix}$$

It is a famous result in differential algebra and even all over mathematics that the  $n$  elements in a differential field are linearly dependent over the field of constants if and only if their Wronskian vanishes.

We will call an extension of the form  $A = K \langle u_1, \dots, u_n \rangle$  with  $u_1, \dots, u_n$  are solutions of

$$(26) \quad L(y) = \frac{W(y, u_1, \dots, u_n)}{W(u_1, \dots, u_n)} = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

a Picard extension. We have the following immediate lemma;

**Lemma 3.1.** [K] *Let  $K \subset L \subset M$  be differential fields. Suppose that  $L$  is Picard over  $K$  and  $M$  has the same field of constants as  $K$ . Then any differential automorphism of  $M$  over  $K$  sends  $L$  into itself.*

**Theorem 3.2.** [K] *The differential Galois group of a Picard extension is an algebraic matrix group over the field of constants.*

The Theorem says that there exists a set  $S$  of polynomials in  $n^2$  ordinary variables such that any differential isomorphism of a Picard extension gives rise to a matrix of constants satisfying  $S$ .

**Theorem 3.3.** [K] *If  $K$  has an algebraically closed constant field of characteristic 0, and  $M$  a Picard extension of  $K$ , then any differential isomorphism over  $K$  between two intermediate fields extends to the whole  $M$ . In particular this also holds for any differential automorphism of an intermediate field over  $K$ .*

In fact over a constant field of  $\text{char} = 0$  any differential isomorphism between intermediate fields extends to the whole differential field.

**Theorem 3.4.** [K] *In the same situation as the previous theorem Galois theory implements a one-to-one correspondence between the intermediate differential fields and the algebraic subgroups of the differential Galois group  $G$ . A closed subgroup  $H$  is normal iff the corresponding field  $L/K$  is normal, then  $G/H$  is the full differential Galois group of  $L$  over  $K$ .*

Let  $A = K \langle u_1, \dots, u_n \rangle$  be a Picard extension and  $W$  the Wronskian of  $u_1, \dots, u_n$ . A basic fact about the Wronskians is that for a differential automorphism  $\sigma$  of  $A$  we have  $\sigma(W) = |c_{ij}|W$ . Then by this one deduces that  $W$  is fixed by  $\sigma$  if and only if  $|c_{ij}| = 1$ .

A family of elements  $(x_i)_{i \in I}$  is called differential algebraic independent; if the family  $(x_i^{(j)})_{i \in I, j \geq 0}$  is algebraically independent over the field of constants. An element  $x$  is called differentially algebraic if the family consisting of  $x$  only is differential algebraic dependent. An extension is called differential algebraic if any element of it, is so. Finally we say  $G$  is differentially finite generated over  $F$  if there exists elements  $x_1, \dots, x_n \in G$  such that  $G$  is generated over  $F$  by the family  $(x_i^{(j)})_{1 \leq i \leq n, j \geq 0}$ .

**Theorem 3.5.** *Let  $F \subset G$  be an extension of differential fields, then*

- *If  $G = F \langle x_1, \dots, x_n \rangle$  and each  $x_i$  is differential algebraic over  $F$  then  $G$  is finitely generated over  $F$ .*
- *If  $G$  is differential finite generated over  $F$  and  $F \subset E \subset G$  is an intermediate differential field, the  $E$  is also differentially finite generated.*

Now we apply the Galois theory to the coordinate ring of fibers of the Demailly-sample bundle. Denote this algebra by  $A_{k,p}$  and its quotient field by  $K_{k,p}$ . It is a finitely generated differential field on the generators  $\xi_1, \dots, \xi_k$ . Then the Lie groups  $G_{k,p}, U_{k,p}, G'_{k,p}$  act as the differential groups of automorphisms and their invariants are the fixed fields of these groups regarded as differential Galois groups. For instance take  $G'_{k,p}$ . According to the proposition 1.3 it can be embedded in  $SL(\oplus_i S^i \mathbb{C}^p)$ . So its fixed field contains the fixed field of  $SL(\oplus_i S^i \mathbb{C}^p)$  which is the algebra generated over  $\mathbb{C}$  by the Wronskians.

**Corollary 3.6.** *In the set up of section 1 all the automorphisms in  $U_k$  extend to  $SL(\text{sym}^{\leq k} \mathbb{C}^p)$ .*

*Proof.* The proof follows from the fact that any element of  $U_k$  is a differential isomorphism of the quotient field  $\text{quot}(\mathcal{O}(J_{k,x}))$ , and the fact that  $U_k \subset SL(\text{sym}^{\leq k} \mathbb{C}^p)$  followed by Theorem 3.3.  $\square$

**Proposition 3.7.** *The algebra  $(\mathcal{O}(J_{k,p})_x)^{G'_{k,p}}$  is finite algebraic over the  $\mathbb{C}$ -algebra generated by the Wronskians.*

*Proof.* This follows from Theorem 3.4, for  $\text{quot}(\mathcal{O}(J_{k,x}))$ . The the theorem follows from the Noether normalization theorem and the explanation before the corollary. In fact Theorem 3.5 shows that all the intermediate fields obtained by the fixed fields of subgroups of  $G_k$  are finitely generated.  $\square$

**Remark 3.8.** *Let  $J = \mathbb{C} \langle \xi_1, \dots, \xi_k \rangle$  be a differential field obtained by adjoining  $n$ -differential indeterminates. Let  $g$  be any non-singular linear transformation*

$$(27) \quad g \cdot \xi_i = \sum c_{ij} x_j$$

*Define  $g$  on all the differential variables by*

$$(28) \quad g \cdot \xi_i^{(m)} = \sum c_{ij} x_j^{(m)}$$



Then  $g$  is a differential automorphism of  $J$ . Define  $L(y)$  by (23). Then

$$(29) \quad L(y) = 0$$

is a linear differential equation which  $\xi_i$  are its independent solutions.  $J$  is a Picard extension of  $\mathbb{C}$  and its differential Galois group is the full linear group,  $[K]$ .

**Remark 3.9.** There are two main type of extensions appearing in a Picard extension  $\mathbb{C} < \xi >$ . One is when the differential Galois group is the additive group  $\mathbb{G}_a$ . This type of extension correspond to solutions of differential equations of the form

$$(30) \quad \xi' = a, \quad a = \text{constant}$$

The second important type is when the Galois group is the multiplicative group  $\mathbb{G}_m$  which correspond to solutions of

$$(31) \quad \xi' - a\xi = a, \quad a = \text{constant}$$

The latter type of extensions are called extension by exponential of an integral,  $[K]$ .

**Remark 3.10.** If  $P = P(f, f', \dots, f^{(k)})$  and  $Q = Q(f, f', \dots, f^{(k)})$  are two local sections of the Green-Griffiths bundle, then the first invariant operator is probably

$$(32) \quad \nabla_j : f \mapsto f'_j$$

Define a bracket operation as follows

$$(33) \quad [P, Q] = \left( d \log \frac{P^{1/\deg(P)}}{Q^{1/\deg(Q)}} \right) \times PQ = \frac{1}{\deg(P)} P dQ - \frac{1}{\deg(Q)} Q dP$$

Later one successively defines the brackets

$$(34) \quad \begin{aligned} [\nabla_j, \nabla_k] &= f'_j f''_k - f''_j f'_k \\ [\nabla_j, [\nabla_k, \nabla_l]] &= f'_j (f'_k f''_l - f''_k f'_l) - 3f'_j (f'_k f''_l - \dots) \end{aligned}$$

If  $(V, h)$  is a Hermitian vector bundle, the equations in (43) define inductively  $G_k$ -equivariant maps

$$(35) \quad Q_k : J_k V \rightarrow S^{k-2} V \otimes \bigwedge^2 V, \quad Q_k(f) = [f', Q_{k-1}(f)]$$

The sections produced by  $Q_k(f)$  generate the fiber rings of Demaily-Semple tower. In fact taking charts on the projective fibers one can check that locally the ring that these sections generate are equal to that in the Proposition 3.6.

**Remark 3.11.** *The geometry of Demaily-Semple tower is very much tied with the Schubert calculus of Wronskians. Consider the linear differential equation*

$$(36) \quad \xi^{(r+1)} - a_1 \xi^{(r)} + \dots + (-1)^{r+1} a_{r+1} \xi = 0$$

where  $a_1, \dots, a_{r+1}$  are indeterminates. Let  $b_j \in \mathbb{C}[a_1, \dots, a_{r+1}]$  be defined via the generating function

$$(37) \quad \frac{1}{1 - a_1 t + \dots + (-1)^{r+1} a_{r+1} t^{r+1}} = \sum_j b_j t^j$$

Then the power series

$$(38) \quad \xi_0 = \sum_n b_n \frac{t^n}{n!}$$

is a solution of (29). This fact basically goes back to the Schubert calculus hidden by Wronskians. Assume  $f = (f_1, \dots, f_r) \in \mathcal{O}_{\mathbb{C}}^r$ . Let

$$(39) \quad \lambda : \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0, \quad |\lambda| = \lambda_0 + \dots + \lambda_r$$

be a partition of weight  $|\lambda|$ . Then one defines the generalized Wronskian associated to  $\lambda$  by

$$(40) \quad W_\lambda(f) = \det(f_i^{(j+\lambda_r-j)})_{0 \leq i, j \leq r}$$

Then one can show that

$$(41) \quad D^{k+1} W(f) = \sum_{|\lambda|=k} c_\lambda W_\lambda(f), \quad c_\lambda = \frac{\lambda!}{k_1 \dots k_{|\lambda|}}$$

where  $k_i$  are the hook length of the boxes of the Young diagram of  $\lambda$ . One can re-write the equation (29) in the form

$$(42) \quad \begin{vmatrix} \xi^{(0)} & \xi^{(1)} & \dots & \xi^{(r+1)} \\ f'_1 & f'_2 & \dots & f'_r \\ f_1^{(r+1)} & & & f_r^{(r+1)} \end{vmatrix} = 0$$

using the above notation

$$(43) \quad W_0(f) \xi^{(r+1)} - W_{(1)}(f) \xi^{(r)} + \dots + (-1)^{r+1} W_{(1^{r+1})}(f) \xi = 0$$

where  $(1)^k = (1, \dots, 1, 0, 0, \dots)$  is the primitive partition. There is a well known formula

$$(44) \quad W_\lambda(f) = \Delta_\lambda(b)W_0(f)$$

namely Giambelli formula ( $b$  is defined by (30)), where

$$(45) \quad \Delta_\lambda(x) = \det(x_{\lambda_r-j+i})$$

is the Schur polynomial corresponding to  $\lambda$ . Using this fact we will write

$$(46) \quad W_{(1^k)}(f) = a_k W(f)$$

which will reach us to (29), so on one property of one solution of one polynomial, [GS].

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CENTRO DE INVESTIGACION EN MATEMATICAS, CIMAT

,

E-mail address: mrahmati@cimat.mx